On Partitions of Finite vector Spaces

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Abstract

In this note, we give a new necessary condition for the existence of non-trivial partitions of a finite vector space. Precisely, we prove that the number of the subspaces of minimum dimension t of a non-trivial partition of $V_n(q)$ is greater than or equal to $q + t$. Moreover, we give some extensions of a well known Beutelspacher-Heden's result on existence of T-partitions.

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1 Introduction

A partition $\mathbf{P}$ of the n-dimensional vector space $V_n(q)$ over the finite field with q elements, is a set of non-zero subspaces (components) of $V_n(q)$ such that each non-zero element of $V_n(q)$ is contained in exactly one element of $\mathbf{P}$.

The interest on the problems of existence, enumeration, and classification of partitions, arises in connection with the construction of codes and combinatorial designs. In fact, if $\mathbf{P} = \{V_1, V_2, ..., V_r\}$ is a non-trivial partition of

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$V_n(q)$, then the subspace W of the vector space $V_1 \times V_2 \times ..... \times V_r$, which is defined by

$$W := \{(y_1, y_2, ..., y_r) \in V_1 \times V_2 \times ..... \times V_r \mid \sum_{i=1}^{r} y_i = \{0\}\},$$

is a perfect mixed linear code (see [HS] and [Li]). Moreover, if $\mathbb{B}$ is the set of subspaces in $\mathbf{P}$ together with all their cosets, then $V_n(q)$ and $\mathbb{B}$ are, respectively, the point set and the block set of an uniformly resolvable design which admits a translation group isomorphic to $V_n(q)$ (see [DR] and [ESSSV2]). Furthermore, combinatorial designs can be associated with certain more general "partitions" (see for instance [Sc1], [Sc2] and [Sp]).

If a partition consists of x_i components of dimension n_i for each $i = 1, 2, ..., k$, then the non-negative integers $x_1, x_2, ..., x_k$ are a solution of the Diophantine equation $\sum_{i=1}^{k}(q^{n_i} - 1)X_i = q^n - 1$. A fundamental problem about partitions of $V_n(q)$ is to give necessary and sufficient conditions on non-negative solutions of the above equation, in order that they correspond to partitions of $V_n(q)$.
In [ESSSV2] the authors gave such conditions in the case where $q = 2$, $k = 2$, $n_1 = 2$ and $n_2 = 3$ (see section 2). In order that, they proved the following two necessary conditions for existence of partitions:
a) In a non-trivial partition of $V_n(q)$ the number of components of minimum dimension must be greater than or equal to 2.
b) If the components of minimum dimension of a partition of $V_n(2)$ have dimension 1, then their number is greater than or equal to 3 (see [ESSSV1]).

Another interesting problem is related to existence results on T-partitions, where, if T is a set of positive integers, a T-partition is a partition $\mathbf{P}$ of $V_n(q)$ such that $\{dimV' \mid V' \in \mathbf{P}\} = T$. Clearly the existence of a T-partition of $V_n(q)$ implies the existence of a positive solution of the equation $\sum_{i=1}^{k}(q^{n_i} - 1)X_i = q^n - 1$ in the case where $T = \{n_1, n_2, ..., n_k\}$. A. Beutelspacher and O. Heden proved (see [Be] and [He]) the existence of a T-partition in the case where $\min T \geq 2$ and $\max T = \frac{n}{2}$.

In this paper, in section 2 we will recall some definitions and some known results about the existence of partitions of a finite vector space. In section 3, we will provide a more general necessary condition on the minimum dimension components of a partition. Precisely, we will shown that the number of components of minimum dimension t, of any non-trivial partition of $V_n(q)$, is greater than or equal to $\alpha q + t$ where α is a positive integer. Finally,

in section 4, we will extend the mentioned Beutelspacher-Heden's existence result for T-partition of $V_n(q)$ in the case where the minimum dimension of the components is 1 and in some other cases where the maximum dimension of the components is consistent with n.

2 Definitions and first results

In this section we recall some basic properties of partitions of finite vector spaces and some results about the existence of certain classes of partitions.

Let n be a positive integer $(n > 1)$, q be a prime power, $\mathbb{F}_q$ be the finite field of order q and $V_n(q)$ be the n-dimensional vector space over $\mathbb{F}_q$. A set $\mathbf{P} = \{V_1, V_2, ..., V_r\}$ of non-zero subspaces of $V_n(q)$ is a partition of $V_n(q)$ if and only if $\cup_{i=1}^{r} V_i = V_n(q)$ and $V_i \cap V_j = \{0\}$ for every $i, j \in \{1, 2, ..., r\}$ and $i \neq j$. We call $\mathbf{P}$ non-trivial in the case where $r \geq 2$. The elements of $\mathbf{P}$ are said to be the components of the partition. Let T be a set of positive integers and $\mathbf{P}$ a set of disjoint non-trivial subspaces (which is not necessarily a partition). $\mathbf{P}$ is said to be a T-set of subspaces (a T-partition if $\mathbf{P}$ is a partition) if the map $dim : \mathbf{P} \to T$ is surjective, where $dim(V_i)$ is the dimension of the subspace V_i for each $V_i \in \mathbf{P}$. Of course if $T = \{n_1, n_2, ..., n_k\}$, then $1 \leq n_i \leq n$ for every $i = 1, 2, ..., k$.
A. Beutelspacher and O. Heden proved (see [Be] and [He]) the following well known existence result.

2.1 Theorem. *Let $T = \{n_1, n_2, ..., n_k\}$ be a set of positive integers with $n_1 < n_2 < ...... < n_k$. If $n_1 \geq 2$, then there exists a T-partition of $V_{2n_k}(q)$.*

The above theorem was proved by Beutelspacher [Be] in the case where $n_1 = 2$ and by Heden [He] for $n_1 > 2$.

Now, let k be a positive integer and $x_1, x_2, ..., x_k$ be non-negative integers. If $\mathbf{P}$ is a partition of $V_n(q)$ which contains x_i components of dimension n_i for each $i = 1, 2, ..., k$, then we say that $\mathbf{P}$ is of type

$$[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$$

or that $\mathbf{P}$ is an $[(x_1,n_1),(x_2,n_2),......,(x_k,n_k)]$-partition of $V_n(q)$ (see [ES-SSV2]). Note that it is possible to have $x_i = 0$ for some $1 \le i \le k$. Clearly, in such a case, there are no components of dimension n_i. However, as we will see, such notation will be useful when we associate partitions to non-negative solutions of some Diophantine equation. Later on, for a partition of type $[(x_1,n_1),(x_2,n_2),......,(x_k,n_k)]$ we will always suppose $1 \le n_1 < n_2 < ...... < n_k \le n$.

Let $\mathbf{P}$ be an $[(x_1,n_1),(x_2,n_2),......,(x_k,n_k)]$-partition of $V_n(q)$. Then it is easy to show the following necessary condition.

1) $(x_1,x_2,......,x_k)$ is a non-negative solution of the Diophantine equation

$$\sum_{i=1}^{k}(q^{n_i}-1)X_i = q^n - 1. \tag{1}$$

Moreover, if V_i and V_j are two distinct components of $\mathbf{P}$, then $dim(V_i+V_j) = dimV_i + dimV_j$ since $V_i \cap V_j = 0$. Hence, the following necessary conditions are obtained:

2) If $i \ne j$ and $x_i \ne 0 \ne x_j$, then $n_i + n_j \le n$.
If $2n_i > n$, then $x_i \le 1$.

Note that, by Theorem 2.1, the equation (1) always admits a non-negative solution when $n = 2n_k$ and $n_1 \ge 2$.

Furthermore, if W is a subspace of $V_n(q)$ of dimension s, then $\mathbf{P'} = \{V_i \cap W \mid V_i \in \mathbf{P}\ and\ V_i \cap W \ne 0\}$ is a partition of W and so, if $\mathbf{P'}$ is of type $[(x'_1,n'_1),(x'_2,n'_2),......,(x'_{k'},n'_{k'})]$, we also have

3) the equation $\sum_{i=1}^{k'}(q^{n'_i}-1)X_i = q^s - 1$ admits the non-negative solution $(x'_1,x'_2,.....,x'_{k'})$.

In [Bu] it was shown that for $s = n-1$ the property 3) is a necessary condition which does not follow from conditions 1) and 2).

Now we recall some existence results. The following two theorems can be found in [Bu], but Theorem 2.2 was known before. In fact, part i) of it is a well known result on d-spreads and part ii) has been also proved previously by Beutelspacher in [Be].

2.2 Theorem. *Let d and n be positive integers.*

i) If d divides n, then there exists a partition of $V_n(q)$ of type $[(\frac{q^n-1}{q^d-1}, d)]$.
ii) If $d < \frac{n}{2}$, then there exists a partition of $V_n(q)$ of type $[(q^{n-d}, d), (1, n-d)]$.

2.3 Theorem. *Let n, k and d be positive integers with $d > 1$. If $n = kd - 1$, then there exists a partition of $V_n(q)$ of type $[(q^{(k-1)d}, d-1), (\frac{q^{(k-1)d}-1}{q^d-1}, d)]$.*

For partitions of finite vector spaces, we have the following fundamental problem.

2.4. *Give necessary and sufficient conditions on non-negative solutions of the Diophantine equation (1) such that they correspond to $[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$-partitions of $V_n(q)$.*

For small value of k some result is available.

2.5 Proposition. *The properties 1) and 2) are necessary and sufficient conditions for the existence of an $[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$-partition of $V_n(q)$ when $k = 1$ or $k = 2$ and $n_1 + n_2 = n$.*

Proof. In the case where $k = 1$, to every solution of (1) corresponds an $[(x_1, n_1)]$-partition. In fact, the equation $(q^{n_1} - 1)x_1 = q^n - 1$ admits a solution if and only if n_1 divides n and so, by i) of Theorem 2.2, there exists an $[(x_1, n_1)]$-partition.
Now consider the case where $k = 2$ and $n_1 + n_2 = n$. Suppose that (x_1, x_2) is a non-negative solution of (1) which also verifies the necessary condition 2). Since $n_2 = n - n_1 > n_1$, we have $n_1 < \frac{n}{2}$ and so $n_2 > \frac{n}{2}$. It follows by 2) that $x_2 \leq 1$. If $x_2 = 0$, then from the equation (1) we obtain that n_1 divides n and so, again by i) of Theorem 2.2, we have a $[(\frac{q^n-1}{q^{n_1}-1}, n_1), (0, n_2)]$-partition. In case $x_2 = 1$, from the equation (1) we get $x_1 = q^{n-n_1}$. Therefore, $(x_1, x_2) = (q^{n-n_1}, 1)$ and so, by ii) of Theorem 2.2, there exists a $[(q^{n-n_1}, n_1), (1, n_2)]$-partition. So the proposition is proved.

For $k = 2$ and for any n_1 and n_2 the question is still an open problem. Recently, in [ESSSV1] and [ESSSV2] the authors resolved it in the case where $n_1 = 2$, $n_2 = 3$ and $q = 2$.

2.6 Theorem. *There exists a partition of $V_n(2)$, $n \geq 3$, of type $[(x_1, 2), (x_2, 3)]$ if and only if (x_1, x_2) is a solution of the Diophantine equation*

$$3x_1 + 7x_2 = 2^n - 1 \tag{2}$$

with x_1 and x_2 non-negative integers and $x_1 \neq 1$.

In order to show Theorem 2.6, they gave the following theorems.

2.7 Theorem. *Let V and V' be $\mathbb{F}_q$-vector spaces of finite dimension and $T = \{n_1, n_2, ..., n_k\}$ a set of positive integers with $n_1 < n_2 < ...... < n_k$. If $\mathbf{P}$ is a T-partition of V with $n_k \leq dimV'$, then there exists a T-set of subspaces $\overline{\mathbf{P}}$ of $V \oplus V'$ such that $|\overline{\mathbf{P}}| = (q^{dimV'} - 1)|\mathbf{P}|$ and $\{V, V'\} \cup \overline{\mathbf{P}}$ is a partition of $V \oplus V'$.*

2.8 Theorem. *Let $\mathbf{P}$ be a non-trivial partition of $V_n(q)$ of type $[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$.*

i) If $x_1 \neq 0$, then $x_1 \geq 2$.

ii) If $n_1 = 1$ and $q = 2$, then $x_1 \geq 3$.

2.9 Remark. Note that Theorem 2.7 is a generalization of Theorem 2.2, part ii). In fact if $d = dimV$, $n - d = dimV'$ with $d < n - d$ and $\mathbf{P} = \{V\}$, then there exists a partition of $V_n(q) = V \oplus V'$ of type $[(q^{n-d}, d), (1, n - d)]$. Moreover note that Theorem 2.8 gives information about the number of minimum dimension components of a partition.

2.10 Remark. Observe that, if $\mathbf{P}$ is a non-trivial $[(x_1, 2), (x_2, 3)]$-partition of $V_n(2)$ with x_1 and x_2 positive integers, then $x_1 \geq 3$. In fact, if $x_1 = 2$, from equation (2), we have $x_2 = \frac{2^n-1-6}{7}$ and so 7 divides 2^n which is a contradiction. Of course, if either x_1 or x_2 is equal to zero, we get $x_1 + x_2 \geq 3$ since a group can not be the union of two proper and disjoint subgroups. Therefore, if $\mathbf{P}$ is a non-trivial $[(x_1, 2), (x_2, 3)]$-partition of $V_n(2)$, then the number of its minimum dimension subspaces is greater than or equal to 3.

3 A new necessary condition

In this section we will prove that, for every prime power q and for every non-trivial partition $\mathbf{P}$ of $V_n(q)$, the number of minimum dimension subspaces in $\mathbf{P}$ is always greater than or equal to $q + t$.

First we need the following lemmas.

3.1 Lemma. *Let* $\mathbf{P} = \{V_1, V_2, ..., V_r\}$ *be a non-trivial partition of* $V_n(q)$ *and* t *and* s *be positive integers with* $t < n$ *and* $s < r$. *If* $dim(V_i) = t$ *for every* $i = 1, ..., s$ *and* $dim(V_j) \geq t + 1$ *for every* $j = s + 1, ..., r$, *then* $s \geq \alpha q$ *for some positive integer* α.

Proof. We can suppose that $V_n(q) = \mathbb{F}_q^n$ and $V_1 = \mathbb{F}_q^t \times \{0\}^{n-t}$ after we choose an ordered basis of $V_n(q)$ which extends a fixed ordered basis of V_1. Let $W = V_1^{\perp}$ be the dual subspace of V_1 with respect to the canonical inner product on $\mathbb{F}_q^n$. Of course we have $V_1 \cap W = \{0\}$ and $V_n(q) = V_1 \oplus W$. Moreover, for every $j = s + 1, ..., r$, we have that $dim(V_j + W) = dim(V_j) + dim(W) - dim(V_j \cap W) = n - t + dim(V_j) - dim(V_j \cap W)$. But $dim(V_j) > dim(V_1) = t$, then $z := dim(V_j) - t \geq 1$. Thus we have $dim(V_j + W) = n + z - dim(V_j \cap W) \leq n$. It follows $dim(V_j \cap W) \geq z \geq 1$ and so we get $V_j \cap W \neq \{0\}$.

Of course it is possible that there are some other subspaces of dimension t, different from V_1, which are not disjoint from W. So, we can suppose that there exists an integer s' with $1 \leq s' \leq s$ and such that $V_i \cap W = \{0\}$ for every $i = 1, ..., s'$, whereas $V_i \cap W \neq \{0\}$ for every $i > s'$ and $i \leq s$. Consider the partition $\mathbf{P}'$ induced by $\mathbf{P}$ on W, that is $\mathbf{P}' = \{V_i \cap W \mid V_i \in \mathbf{P} \textit{ and } V_i \cap W \neq \{0\}\}$. Clearly we have

$$\sum_{j=s'+1}^{r} (q^{m_j} - 1) = q^{n-t} - 1, \tag{4}$$

where $m_j = dim(V_j \cap W) \geq 1$ for every $j = s' + 1, ..., r$. Further, considering the partition $\mathbf{P}$, we obtain $\sum_{j=s'+1}^{r} (q^{n_j} - 1) = q^n - 1 - \sum_{j=1}^{s'} (q^{n_j} - 1) = q^n - 1 - s'(q^t - 1)$, from which

$$\sum_{j=s'+1}^{r} (q^{n_j} - 1) = q^n - 1 - s'q^t + s'. \tag{5}$$

Now, subtracting (4) from (5), we get

$$\sum_{j=s'+1}^{r} (q^{n_j} - q^{m_j}) = q^n - s'q^t - q^{n-t} + s'.$$

Hence we obtain that q divides s'. But $s' \neq 0$ being $V_1 \cap W = \{0\}$. Therefore, for some positive integer α, we obtain $s \geq s' = \alpha q$ and the proof is complete.

3.2 Lemma. *Let* $\mathbf{P}$ *be a non-trivial partition of* $V_n(q)$ *of type* $[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$. *Suppose* i_0 *be a positive integer such that* $x_{i_0} \neq 0$ *and* $x_i = 0$ *for each* $i = 1, 2, ..., i_0 - 1$, *then* $x_{i_0} \geq \alpha q + 1$ *where* α *is a positive integer.*

Proof. We use the same notations as in the previous lemma. If $i_0 = k$, then we have that $x_{i_0} = x_k = \frac{q^n-1}{q^{n_k}-1}$ because of condition 1). So n_k divides n. Moreover, being $\mathbf{P}$ a non-trivial partition, we have $n_k < n$. It follows that $x_{i_0} = q^{n-n_k} + q^{n-2n_k} + ... + q^{n-(\frac{n}{n_k}-1)n_k} + 1 = \alpha q^{n_k} + 1 \geq \alpha q + 1$ where $\alpha = q^{n-2n_k} + q^{n-3n_k} + ... + q^{n_k} + 1$.
Now suppose $i_0 < k$ and set $t = n_{i_0}$ and $s = x_{i_0}$ as in the previous lemma. If $\mathbf{P} = \{V_1, V_2, ..., V_r\}$, and $V_1, V_2, ..., V_s$ are the s components of dimension t, then there exist at least two distinct components which have the same minimum dimension t since, by Lemma 3.1, $s \geq q \geq 2$. Therefore, we certainly have that V_1 is distinct from V_s. So $dim(V_1 + V_s) = 2t \leq n$ because of V_1 and V_s belong to $\mathbf{P}$ and $dim(V_1) = dim(V_s) = t$. Let $\{v_1, v_2, ..., v_t\}$ be an ordered basis of V_1 and $\{v'_1, v'_2, ..., v'_t\}$ be an ordered basis of V_s. Then the vectors $\{v_1, v_2, ..., v_t, v'_1, v'_2, ..., v'_t\}$ are a basis of $V_1 + V_s$. So they are linearly independent and we can consider a basis of $V_n(q)$ which contains them. In relation to this new basis we can identify $V_n(q)$ with $\mathbb{F}_q^n$, V_1 with $\mathbb{F}_q^t \times \{0\}^{n-t}$ and V_s with $\{0\}^t \times \mathbb{F}_q^t \times \{0\}^{n-2t}$. Let W be the dual space of V_1, that is $W = \{0\}^t \times \mathbb{F}_q^{n-t}$. It follows that $V_s \subseteq W$ and so $V_s \cap W \neq \{0\}$. Therefore, the number s' of the t-dimensional subspaces of $V_n(q)$ which are disjoint from W is smaller than s. But, as in the proof of Lemma 3.1, we have $s' = \alpha q$ for some positive integer α. So $s > s' = \alpha q$, that is to say $x_{i_0} = s \geq \alpha q + 1$, and the lemma is shown.

3.3 Theorem. *In any non-trivial partition of* $V_n(q)$, *the number of subspaces of minimum dimension* t *is greater than or equal to* $\alpha q + t$ *for some positive integer* α.

Proof. We proceed by induction on t. For $t = 1$ the theorem is true by the above Lemma 3.2. Now, let $t \geq 2$, $\mathbf{P}$ be a partition of $V_n(q)$ and S be the subset of $\mathbf{P}$ of all the components of minimum dimension t. Consider an hyperplane W of $V_n(q)$ which contains at least a component of S and it does not contain all the contains of S. A such hyperplane there exists since $s = |S| \geq \alpha q + 1 \geq q + 1 \geq 3$ by the above lemma. Being $t > 1$ the partition $\mathbf{P}_W$, which is induced from $\mathbf{P}$ on W, has components of minimum dimension $t - 1$. Let S' be the set of such components of $\mathbf{P}_W$. So, by induction, their number is $s' = |S'| = \alpha q + t - 1$. But if $V_i' \in S'$, then $V_i' = V_i \cap W$ where $V_i \in S$. Therefore, $s \geq s' = \alpha q + t - 1$. Moreover, by construction, W contains at least one component of S. Thus we get that $s \geq s' + 1 \geq (\alpha q + t - 1) + 1$. So $s \geq \alpha q + t$ and the proof is complete.

Finally we can state the next corollary which clearly follows from the above theorem.

3.4 Corollary. *Let $V_n(q)$ be a vector space which admits a non trivial partition $\mathbf{P}$. Then the number of components of $\mathbf{P}$ of minimum dimension t is greater than or equal to $q + t$.*

Now we observe that, if $\mathbf{P} = \{V_1, V_2, ..., V_r\}$ is a non-trivial partition of $V_n(q)$ whose components are all of the same dimension t, then r is equal to the number s of minimum dimension components of $\mathbf{P}$ and t divides n by Proposition 2.5. So we have that $r = s = \frac{q^n-1}{q^t-1} \geq q^t + 1$ (Note that, s may be much greater than $q + t$ if $t \neq 1$). More generally, we have the following proposition.

3.5 Proposition. *Let $\mathbf{P}$ be a non-trivial partition of $V_n(q)$ which have r components. If t is the minimum dimension of the components of $\mathbf{P}$, then $q^t + 1 \leq r \leq \lfloor \frac{q^n-1}{q^t-1} \rfloor$.*
(Here $\lfloor x \rfloor$ denotes the integer part of the real number x) .

Proof. Suppose $\mathbf{P} = \{V_1, V_2, ..., V_r\}$ be the partition of $V_n(q)$. Then we have $\sum_{i=1}^{r}(q^{n_i} - 1) = q^n - 1$ if n_i denotes the dimension of V_i for every $1 \leq i \leq r$. Hence $r - 1 = \sum_{i=1}^{r} q^{n_i} - q^n = q^t(\sum_{i=1}^{r} q^{n_i - t} - q^{n-t})$ and we obtain that $r = \alpha q^t + 1$ with $\alpha \geq 1$ being $\mathbf{P}$ a non-trivial partition. It follows that $r \geq q^t + 1$. Now, let $\{V_1, V_2, ..., V_s\}$ be the components of $\mathbf{P}$ of minimum dimension t and suppose $s < r$. We have that $(V \backslash \{0\}) \backslash (\bigcup_{i=1}^{s}(V_i \backslash \{0\})) = \bigcup_{i=s+1}^{r}(V_i \backslash \{0\})$. So we obtain $|\bigcup_{i=s+1}^{r}(V_i \backslash \{0\})| = |V \backslash \{0\}| - |\bigcup_{i=1}^{s}(V_i \backslash \{0\})|$.

But $(r-s)(q^t-1) \ < |\bigcup_{i=s+1}^{r}(V_i\backslash\{0\})|$ since $|V_i| > q^t$ for every $s+1 \leq i \leq r$. It follows that $(r-s)(q^t-1) < |V\backslash\{0\}|-|\bigcup_{i=1}^{s}(V_i\backslash\{0\})| = (q^n-1)-s(q^t-1)$ and so $(r-s)(q^t-1) < q^n-1-sq^t-s$ from which we get $r < \frac{q^n-1}{q^t-1}$. If $s = r$, then the components of $\mathbf{P}$ have all the same dimension t and, as observed before, $r = \frac{q^n-1}{q^t-1}$. Therefore, in any case, $r \leq \frac{q^n-1}{q^t-1}$ and the proposition is shown.

3.6 Remark. Proposition 3.5 and the examples of partitions which are known to us, drive us to think that Corollary 3.4 can be substantially improved. In fact, we conjecture that the number of components of minimum dimension t of a non-trivial partition of $V_n(q)$ is greater or equal to $q^t + 1$.

4 Existence results on T-partitions

In this section we give some extensions of Theorem 2.1. To begin, we can drop the hypothesis "$n_1 \geq 2$" in Theorem 2.1. In fact, we have the following proposition.

4.1 Proposition. *Let $T = \{n_1, n_2, ..., n_k\}$ be a set of positive integers such that $n_1 < n_2 < ...... < n_k$. Then there exists a T-partition of $V_{2n_k}(q)$.*

Proof. By Theorem 2.1 we can suppose that $n_1 = 1$. If $k = 1$ the proposition follows by i) of Theorem 2.2. So let $k \geq 2$ and consider the subset $T' = \{n_2, n_3......, n_k\}$ of T. Again by Theorem 2.1, there exists a T'-partition $\mathbf{P}'$ of $V_{2n_k}(q)$ because $n_2 > n_1 = 1$. But, by Corollary 3.4, there exist at least $q + n_2 \geq 4$ components of $\mathbf{P}'$ of minimum dimension n_2. Let V' be such a component of dimension n_2 and consider the partition $\mathbf{P}''$ of V' whose components are all its subspaces of dimension 1. Now, $\mathbf{P} = (\mathbf{P}' \setminus \{V'\}) \cup \mathbf{P}''$ is a T-partition of $V_{2n_k}(q)$ since $|\mathbf{P}''| \geq q + 1 \geq 1$ and there are some other components (at least 3) of $\mathbf{P}' \subset \mathbf{P}$ of dimension n_2. This complete the proof.

4.2 Theorem. *Let $T = \{n_1, n_2, ..., n_k\}$ be a set of positive integers such that $n_1 < n_2 < .... < n_k$ and consider the vector space $V_n(q)$ with $n \geq 2n_k$.*

If $gcd(n, 2n_k)$ admits some divisor into T, then there exists a T-partition of $V_n(q)$.

Proof. By Proposition 4.1 we can suppose that $n > 2n_k$. Consider a subspace V of $V_n(q)$ of dimension $2n_k$ and such that $V \cap V^{\perp} = \{0\}$ where $V^{\perp}$ is the dual space of V. If $n_{i_0} \in T$ is a divisor of $gcd(n, 2n_k)$, then n_{i_0} is a divisor of $n - 2n_k = dimV^{\perp}$. So, by Theorem 2.2, there exists a partition $\mathbf{P}'$ of $V^{\perp}$ whose components have all the same dimension n_{i_0}. Since n_{i_0} divides $2n_k$, for the same reason we get that there exists a partition $\mathbf{P}$ of V whose components have all the same dimension n_{i_0}, that is, $\mathbf{P}$ is a $\bar{T}$-partition of V where $\bar{T} = \{n_{i_0}\}$ and $n_{i_0} \leq dimV^{\perp} = n - 2n_k$ being n_{i_0} a divisor of $n - 2n_k$. It follows, by Theorem 2.7, that there exists a $\bar{T}$-set of n_{i_0}-dimensional subspaces $\mathbf{P}''$ of $V \oplus V^{\perp} = V_n(q)$ such that $\{V, V^{\perp}\} \cup \mathbf{P}''$ is a partition of $V_n(q)$. Now, by the above proposition, let $\bar{\mathbf{P}}$ be a T-partition of V. Then $\bar{\mathbf{P}} \cup \mathbf{P}' \cup \mathbf{P}''$ is a T-partition of $V_n(q)$ and the proof is complete.

Note that the above theorem is Proposition 4.1 for $n = 2n_k$. So it is a generalization of Beutelspacher-Heden's Theorem 2.1.

4.3 Lemma. *Let T be as in Theorem 4.2 and $V_n(q)$ be a vector space over $\mathbb{F}_q$ of dimension $n \geq 3n_k$. If there exists a subset T' of T such that $V_{n-2n_k}(q)$ has a T'-partition, then there exists a T-partition of $V_n(q)$.*

Proof. Let V be a subspace of $V_n(q)$ of dimension $2n_k$ and such that $V \cap V^{\perp} = \{0\}$. Since $dimV^{\perp} = n - 2n_k$, then $V^{\perp}$ is (isomorphic to) $V_{n-2n_k}(q)$ and so $V^{\perp}$ admits a T'-partition $\mathbf{P}'$. By Proposition 4.1, we can consider a T-partition $\mathbf{P}$ of V. Moreover, by hypothesis, $n - 2n_k \geq n_k$ and so $dimV^{\perp} \geq n_k$. Therefore there exists a T-set of subspaces $\mathbf{P}''$ of $V \oplus V^{\perp}$ such that $\{V, V^{\perp}\} \cup \mathbf{P}''$ is a partition of $V \oplus V^{\perp}$. Now we get that $\mathbf{P} \cup \mathbf{P}' \cup \mathbf{P}''$ is a T-partition of $V \oplus V^{\perp} = V_n(q)$ and the lemma is shown.

For n greater than or equal to $3n_k$ or for n smaller than $2n_k$, we have the next theorem.

4.4 Theorem. *Let n be a positive integer and $T = \{n_1, n_2, ..., n_k\}$ be a set of positive integers such that $n_1 < n_2 < .... < n_k < n$. Then $V_n(q)$ admits a T-partition if one of the following hypothesis is satisfied:*

a) $3n_k \leq n$ and $n - 2n_k$ admits some divisor into T;

b) $2n_k > n = n_k + n_{k-1}$ and $n_1 = 1$;

c) $2n_k > n \geq n_k + 2n_{k-1}$ *and* $gcd(n, 2n_{k-1})$ *has some divisor into* T.

Proof. a) By the above lemma it is enough to note that there exists $T' \subseteq T$ such that $V_{n-2n_k}(q)$ has a T'-partition. In fact, if $n_{i_0} \in T$ is a divisor of $n - 2n_k$, then $V_{n-2n_k}(q)$ admits a partition whose components have all the same dimension n_{i_0}. That is to say, $V_{n-2n_k}(q)$ has a T'-partition if we set $T' = \{n_{i_0}\}$.

b) Note that, since $n_k > \frac{n}{2}$, if $V_n(q)$ admits a T-partition then exists exactly one subspace of dimension n_k because of the necessary condition 2). Again we note that if d is a positive integer smaller than the dimension of a vector space U, then always there exists a $\bar{T}$-partition of U where $\bar{T} = \{1, d\}$; in fact, the components of a such $\bar{T}$-partition may be a fixed d-dimensional subspace W of U and the 1-dimensional subspaces which are not in W.
Now we can prove b). By ii) of Theorem 2.2, let $\mathbf{P'}$ be a T'-partition of $V_n(q)$ of type $[(q^{n_k}, n - n_k), (1, n_k)]$. Of course here $T' = \{n - n_k, n_k\}$. Since $q^{n_k} \geq n_k \geq k > k-2$, we can choose $k-2$ distinct subspaces $V_1, V_2, ..., V_{k-2}$ of dimension $n - n_k = n_{k-1}$ of the T'-partition $\mathbf{P'}$. For every $1 \leq i \leq k-2$, being $n_i < n_{k-1} = n - n_k = dimV_i$, it is possible to consider a T_i'-partition $\mathbf{P_i'}$ of V_i where $T_i' = \{1, n_i\}$. Let $\mathbf{P''} = \{V_i \in \mathbf{P'} \mid k-1 \leq i \leq q^{n_k} \textit{ and } dimV_i = n_{k-1}\}$. Since $q^{n_k} > k-2$, we get that $\mathbf{P''} \neq \emptyset$. Now, if V is the component of dimension n_k in $\mathbf{P'}$, then $\{V\} \cup (\cup_{i=1}^{k-2} \mathbf{P_i'}) \cup \mathbf{P''}$ is a T-partition of $V_n(q)$.

c) Let V be a subspace of $V_n(q)$ of dimension $n - n_k$ and such that $V \cap V^\perp = \{0\}$. The hypothesis $n > n_k > \frac{n}{2}$ implies that n_k is not a divisor of $gcd(n, 2n_{k-1})$ and so $gcd(n, 2n_{k-1})$ admits a divisor in $T' = \{n_1, n_2, ..., n_{k-1}\}$. Now since $n - n_k \geq 2n_{k-1}$, by Theorem 4.2, we have that there exists a T'-partition $\mathbf{P'}$ of V. The subspace $V^\perp$ has dimension n_k and so $n_{k-1} < dimV^\perp$. Therefore, by Theorem 2.7, we obtain that there exists a T'-set of subspaces $\mathbf{P''}$ such that $\{V, V^\perp\} \cup \mathbf{P''}$ is a partition of $V_n(q)$. Now we get that $\mathbf{P'} \cup \mathbf{P''} \cup \{V^\perp\}$ is a T-partition of $V_n(q)$. So the theorem is completely shown.

4.5 Remark. Of course the above results do not give a complete answer to the problem to give sufficient conditions for the existence of a T-partition of $V_n(q)$. For example, it is not known if there are T-partitions of $V_n(q)$ when $3n_k > n > 2n_k$ and no integer belonging to T divides $gcd(n, 2n_k)$.

More generally the problem to give necessary and sufficient conditions about a set of positive integers T to have a T-partition of $V_n(q)$ is an open problem. Of course, it is in correlation with the analogous problem on

$[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$-partitions. But, now the problem is to give conditions on the elements of T such that there exists some $[(x_1, n_1), (x_2, n_2), ..., (x_k, n_k)]$-partition where $x_1, x_2, ..., x_k$ are positive integers. Note that, as it follows from the results of this section, it is not necessary to know the positive integers $x_1, x_2, ..., x_k$ to have the existence of a T-partition.

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